Exploring Common Fixed-Point Theorems on Bi-Complex -Valued Fuzzy Metrics

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Abstract: The concept of a bicomplex valued fuzzy metric space is presented in this study, along with an examination of the function of occasionally weakly compatible mappings in the structure of such spaces. We enhance previous results in this field by establishing a number of common fixed-point results. We also provide examples to show that the results acquired are valid. These results advance our knowledge of fixed-point theory in fuzzy experiments and open up new avenues for investigation into generalized metric spaces and their applications.

Keywords: Complex fuzzy set, Bicomplex valued fuzzy metric space, Complex valued continuous t-norms, cf- Point, occasionally weakly compatible mapping.

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I. INTRODUCTION

The concept of a fuzzy set was initially put forth by Zadeh in 1965 [18]. In order to address instances when the data is unclear or imprecise, it assigns a degree to which a certain object belongs to a fuzzy set. In 1975, Kramosil and Michalek [14] presented the innovative idea of FMS in two different methods. George and Veeramani [9] modified the idea of FMS by using continuous t-norms. Later, C.T. Aage and J.N. Salunke [1] developed the notions of weakly compatible mappings, compatible maps in the context of fuzzy metric space, and the proving of several fixed point theorems. After this, Thagafi and Shahzad [2] introduced the concept of sometimes weakly compatible mappings. Later, Pant [19] proposed the concept of reciprocally continuous maps and proved certain common fixed point theorems. Many scholars in this field have produced similar fixed point results that satisfy different commutative conditions. Fuzzy complex numbers and fuzzy complex analysis were initially introduced by Buckly [6].Buckey's study motivated some researchers to continue their research on fuzzy complex numbers. Ramot et al. [17, 18] extended fuzzy sets to complex fuzzy sets in this series as a generalization. Ramot argues that the complex fuzzy set is defined by a membership function that goes beyond [0, 1] to the unit circle in the complex plane. Azam et al. [4] introduced the idea of complex valued metric space and provided sufficient requirements for the existence of a pair of mappings' cf-point that satisfy contractive constraints. Later, D. Singh and Kumam [25] introduced the idea of complex valued fuzzy metric spaces utilizing the complex valued continuous t-norm and related topologies. I. Demir [8] presented the concept of complex valued fuzzy b-metric spaces after examining several fp outcomes. Rakesh Tiwari et al. [27] presented bicomplex valued fuzzy metric space and bicomplex valued fuzzy b-metric space in 2022 and presented some cf-point results.

In this research, we use occasionally weakly compatible mappings to establish some cf-point facts, motivated and inspired by the work of Rakesh Tiwari et al. [27].

II. PRELIMINARIES

In this section, we give some basic definitions which are useful for main result in this paper.

Let $€_1$ be the set of complex numbers and $z_1, z_2 \in €_1$. Define a partial order $\leq$ on $€_1$ as $z_1 \leq z_2$ if $Re(z_1) \leq Re(z_2), Img(z_1) \leq Img(z_2)$. It follows that $z_1 \leq z_2$. If one of the following conditions are holds:

(i) $Re(z_1) = Re(z_2)$ and $Img(z_1) = Img(z_2)$
(ii) $Re(z_1) < Re(z_2)$ and $Img(z_1) = Img(z_2)$
(iii) $Re(z_1) = Re(z_2)$ and $Img(z_1) < Img(z_2)$
(iv) $Re(z_1) < Re(z_2)$ and $Img(z_1) < Img(z_2)$

We write $z_1 \leq z_2$ if $z_1 \neq z_2$ and one of (ii) and (iii) are satisfied and we write $z_1 < z_2$ if only (iv) is satisfied.

We denote the symbol $€_0$,$€_1$, and $€_2$ as a set of real, complex and bicomplex numbers respectively. The set of bicomplex numbers defined as:

$$€_2 = \{w: w = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in €_0\}$$

$$€_2 = \{w: w = z_1 + z_2 i_2 \ z_1, z_2 \in €_1\}$$

Where $z_1 = a_1 + a_2 i_1$, $z_2 = a_3 + a_4 i_1$ and i_1 , i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The product of all units are commutative and satisfy:

$$i_1 i_2 = j, \quad i_1 j = -i_2, \quad i_2 j = -i_1.$$

The inverse of $u = u_1 + i_2 u_2$ exits if $u_1^2 + u_2^2 \neq 0$ i.e $|u_1^2 + u_2^2| \neq 0$ and it is defined as $u^{-1} = \frac{1}{u} = \frac{u_1 - i_2 u_2}{u_1^2 + u_2^2}$, and then u is called invertible.

For a bicomplex number $w = z_1 + z_2 i_2$, the norm is denoted by $\|z_1 + z_2 i_2\|$ and defined by

$$(|z_1|^2 + |z_2|^2)^{1/2} = (|z_1 - z_2 i_1|^2 + |z_1 + z_2 i_1|^2)^{1/2}.$$

If we take $w = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2$ then the norm of w is defined by

$$\|w\| = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^{1/2}$$

The partial order relation $\precsim_{i_2}$ on $€_2$ was defined by Demir et. al. [8] as $\tau \precsim_{i_2} v$ if and only if $z_1 \leq w_1$ and $z_2 \leq w_2$, for $\tau = z_1 + z_2 i_2$ and $v = w_1 + w_2 i_2$ be two bicomplex numbers. It follows that $\tau \precsim_{i_2} v$ if one of the following conditions is satisfied:

(i) $z_1 = w_1$, $z_2 = w_2$,

(ii) $z_1 \prec w_1$, $z_2 = w_2$,

(iii) $z_1 = w_1$, $z_2 \prec w_2$,

(iv) $z_1 \prec w_1$, $z_2 \prec w_2$,

We write $\tau \lnsim_{i_2} v$ if $\tau \precsim_{i_2} v$ and $\tau \neq v$ and one of (ii) , (iii) and (iv) are satisfied and we will write $\tau \prec v$ if only (iv) is satisfied.

Definition 2.1: A map $*: [0,1] \times [0,1] \to [0,1]$ is called continuous triangular norm, if it's satisfied the following circumstance for $a, b, c, d \in [0,1]$:

(i) $a * b = b * a$ (Symmetry);
(ii) $a * (b * c) = (a * b) * c$ (Associativity);
(iii) $a * 1 = a$ (boundary condition);
(iv) $a * b \leq c * d$ if $a \leq c$ and $b \leq d$ (Monotonicity);

Definition 2.2[1]: The 3-tuple $(X, \mathcal{M}, *)$ is called FMS if X is an arbitrary set, $*$ is t-norm and $\mathcal{M}$ is a fuzzy set on $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$ and $p, q \geq 0$ then

(FM − 1) $\mathcal{M}(x, y, 0) = 0$;
(FM − 2) $\mathcal{M}(x, y, t) = 1$, $\forall t > 0$ if and only if $x = y$;
(FM − 3) $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;
(FM − 4) $\mathcal{M}(x, y, p) * \mathcal{M}(y, z, q) \leq \mathcal{M}(x, z, p + q)$;
(FM − 5) $\mathcal{M}(x, y, .): (0, \infty) \to [0,1]$ is continuous.

Example 1: Let (X, d) be a metric space, define $u * v = min\{u, v\}$(or $u * v = uv$) for all $u, v \in [0,1]$, define as $\mathcal{M}(u, v, t) = \frac{t}{t + d(x,y)}$ for all $x, y \in X$ and $t > 0$;

Example 2: Let $X = [0, \infty)$, $u * v = uv$ for every, $u, v \in [0,1]$ and d is usual metric defined on X. Define a function $\mathcal{M}(x, y, t) = e^{-\frac{d(x,y)}{t}}$; $x, y, t \in X, t > 0$ then $(X, \mathcal{M}, *)$ is a FMS.

Note: $(M_{Fs} - 4')$. $\mathcal{M}(x, z, max\{p, q\}) \geq \mathcal{M}(x, y, p) * \mathcal{M}(y, z, q)$; if the condition $(M_{Fs} - 4)$ of definition (2.2) is replace by $(M_{Fs} - 4')$. then FMS $(X, \mathcal{M}, *)$ is called non-Archimedean FMS. All non-Archimedean FMSs are FMSs as well.

Definition 2.3[25]: A binary operation $*: r_s e^{i\theta} \times r_s e^{i\theta} \to r_s e^{i\theta}$, where $r_s \in [0,1]$, $\theta \in [0, \frac{\pi}{2}]$ is called complex valued triangular norm (or t-norm) if it satisfies the following properties:

(a) $*$ is associative and commutative.

(b) $*$ is continuous.

(c) $\alpha * e^{i\theta} = \alpha, for\ all\ \alpha \in e^{i\theta}$.

(d) $a * b \lesssim_{i_2} c * d$ $whenever\ a \leq c\ and\ b \leq d, for\ all\ a, b, c, d \in e^{i\theta}$.

Definition 2.4[25]: A complex valued fuzzy metric Space is a triplet $(X, \mathcal{M}, *)$ Where X is non-empty set, * is a complex valued continuous is t-norm and $\mathcal{M}: X^2 \times (0, \infty) \to r_s e^{i\theta}$ is complex valued fuzzy set, satisfying the following properties:

$(M_{CFS} - 1)$ $\mathcal{M}(x, y, t) \succ_{i_2} 0$;

$(M_{CFS} - 2)$ $\mathcal{M}(x, y, t) = e^{i\theta}, for\ all\ t > 0\ iff\ x = y$;

$(M_{CFS} - 3)$ $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;

$(M_{CFS} - 4)$ $\mathcal{M}(x, y, p) * \mathcal{M}(y, z, q) \gtrsim_i \mathcal{M}(x, z, p + q)$;

$(M_{CFS} - 5)$ $\mathcal{M}(x, y, .): (0, \infty) \to e^{i\theta}$ is continuous;

$For\ all\ x, y, z \in X, p, q > 0, r_s \in [0,1], \theta \in [0, \frac{\pi}{2}]$.

Then $(X, \mathcal{M}, *)$ is called $complex\ valued\ fuzzy\ metric\ space$.

Example 3: (X, d) be a metric space, define $u * v = min\{u, v\}(or\ \ u * v = uv)$.

$For\ each\ t > 0, x, y \in X, define\ \mathcal{M}(x, y, t) = e^{i\theta} \frac{ht^n}{ht^n + md(x,y)}, h, m, n \in N$.

If $h = m = n = 1$ then $\mathcal{M}(x, y, t) = e^{i\theta} \frac{t}{t + d(x,y)}$

This complex valued fuzzy metric space induced by a metric did referred to as a standard complex valued fuzzy metric space.

Example 4: (X, d) be a metric space, define $u * v = max\{u + v - e^{i\theta}, 0\}$, for a $fix\ \theta \in \left[0, \frac{\pi}{2}\right]$.

Definition 2.5[27]: A binary operation $*: r_s(1 + i_2)e^{i_1\theta} \times r_s(1 + i_2)e^{i_1\theta} \to r_s(1 + i_2)e^{i_1\theta}$, where $r_s \in [0,1]$, $\theta \in [0, \frac{\pi}{2}]$ is called bicomplex valued triangular norm (or t-norm) if it satisfies the following properties:

(a) $*$ is associative and commutative.

(b) $*$ is continuous.

(c) $\alpha * (1 + i_2)e^{i_1\theta} = \alpha, for\ all\ \alpha \in (1 + i_2)e^{i_1\theta}$.

(d) $a * b \lesssim_{i_2} c * d$ $whenever\ a \leq c\ and\ b \leq d, for\ all\ a, b, c, d \in r_s(1 + i_2)e^{i_1\theta}$.

Definition 2.6[27]: A bicomplex valued fuzzy metric space is a triplet $(X, \mathcal{M}, *)$ Where X is non-empty set, * is a bicomplex valued continuous is t-norm and $\mathcal{M}: X^2 \times (0, \infty) \to r_s(1 + i_2)e^{i_1\theta}$ is bicomplex valued fuzzy set, satisfying the following properties:

$(M_{BCFS} - 1)$ $\mathcal{M}(x, y, t) \succ_{i_2} 0$;

$(M_{BCFS} - 2)$ $\mathcal{M}(x, y, t) = (1 + i_2)e^{i_1\theta}, for\ all\ t > 0\ iff\ x = y$;

$(M_{BCFS} - 3)$ $\mathcal{M}(x, y, t) = \mathcal{M}(y, x, t)$;

$(M_{BCFS} - 4)$ $\mathcal{M}(x, y, p) * \mathcal{M}(y, z, q) \gtrsim_{i_2} \mathcal{M}(x, z, p + q)$;

$(M_{BCFS} - 5)$ $\mathcal{M}(x, y, .): (0, \infty) \to r_s(1 + i_2)e^{i_1\theta}$) is continuous;

$For\ all\ x, y, z \in X, p, q > 0, r_s \in [0,1], \theta \in [0, \frac{\pi}{2}]$.

Then $(X, \mathcal{M}, *)$ is called $bicomplex\ valued\ fuzzy\ metric\ spaces$.

Example 5[27]: let $X = R$, define $u * v = min\{u, v\}, for\ all\ u, v \in r_s(1 + i_2)e^{i_1\theta}$, Where $r_s \in [0,1]$, $\theta \in [0, \frac{\pi}{2}]$ and $k: R^+ \to (0, \infty)$ is non-decreasing continuous function $\mathcal{M}(u, v, t) = r_s(1 + i_2)e^{i_1\theta} - \frac{|u-v|}{k(t)}$, for all $u, v \in X$ and $t \in (0, \infty)$ then $(X, \mathcal{M}, *)$ is a $bicomplex\ valued\ fuzzy\ metric\ spaces$.

Example 6[27]: let $X = N$, define $u * v = max\{u + v - (1 + i_2)e^{i_1\theta}\}$, for all $u, v \in r_s(1 + i_2)e^{i_1\theta}$, Where $r_s \in [0,1]$, $\theta \in [0, \frac{\pi}{2}]$ and

$$\mathcal{M}(u, v, t) = \begin{cases} (1 + i_2)e^{i_1\theta}, if\ u = v; \\ tuv(1 + i_2)e^{i_1\theta}, if\ u \neq v\ and\ t \leq 1; \\ uv(1 + i_2)e^{i_1\theta}, if\ u \neq v\ and\ t > 1; \end{cases}$$

a for all $u, v \in X$ and $t \in (0, \infty)$ then $(X, \mathcal{M}, *)$ is a $bicomplex\ valued\ fuzzy\ metric\ spaces$.

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Definition 2.7: Let $(X, \mathcal{M}, *)$ be a bicomplex valued fuzzy metric space then

(i) A sequence $\{x_n\}$ in X is convergent to a point $x_n \to x$ and if and only if $t > 0, \mathcal{M}(x_n, x, t) \to (1 + i_2)e^{i_1\theta}$ or $|\mathcal{M}(x_n, x, t)| \to 1$ as $n \to \infty$.

(ii) A sequence $\{x_n\}$ in X is called Cauchy sequence if and only if $t > 0, \mathcal{M}(x_m, x_n, t) \to (1 + i_2)e^{i_1\theta}$ or $|\mathcal{M}(x_m, x_n, t)| \to 1$ as $m, n \to \infty$.

(iii) Every Cauchy Sequence in X converges to a point of it.

Definition 2.8: Let X be a set, F and G be a self maps of X then a point $x \in X$ is called a coincidence point of F and G if and only if $Fx = Gx$. We shall call $w = Fx = Gx$ a point of coincidence of F and G.

Definition 2.9: Two maps F and G are said to weakly compatible if they commute at their coincidence points i.e if $Fz = Gz$ some $z \in X$ then $FGz = GFz$.

Example 7: Let R be the set of real numbers with bicomplex valued fuzzy metric spaces. Define $F, G, R \to R$ by $F(x) = 3x$ and $G(x) = x2$ for all $x \in R$ then $F(x) = G(x)$ for $x = 0, 3$ but $FG(0) = GF(0)$ and $FG(3) \neq GF(3)$. F and G are occasionally weakly compatible self maps but not weakly compatible.

Definition 2.10: Let X be a set, F and G be a occasionally weakly compatible self maps of X. if F and G have a unique of coincidence i.e ,$w = Fx = Gx$ then w is the unique common fixed point of F and G.

Lemma 2.11: A bicomplex valued fuzzy metric space $(X, M, *)$ with $\lim_{t\to\infty} M(x, y, t) = (1 + i_2)e^{i_1\theta}$, for all $x, y \in X$. if there exists a constant $h \in (0,1)$ such that $M(x, y, ht) \gtrsim_{i_2} M(x, y, t)$ for all $t > 0$ then $x = y$.

Lemma 2.12: let $\{x_n\}$ be a sequence in a bicomplex valued fuzzy metric space $(X, M, *)$ with $\lim_{t\to\infty} M(x, y, t) = (1 + i_2)e^{i_1\theta}$, for all $x, y \in X$. if there exists a constant $h \in (0,1)$ such that $M(x_{n+1}, x_{n+2}, ht) \gtrsim_{i_2} M(x_n, x_{n+1}, t)$ for all $t > 0$ and $n = 0,1,2 \ldots$ then $\{x_n\}$ is a Cauchy sequence.

Main Result

Theorem (3.1): A bicomplex valued fuzzy metric space $(X, M, *)$ with $\lim_{t\to\infty} M(x, y, t) = (1 + i_2)e^{i_1\theta}$. let $P: X \to X$ be a mapping satisfying if there exist $h \in (0,1)$ such that

$$M(Px, Py, ht) \gtrsim_{i_2} \Omega\left[Min\left\{\begin{matrix} M(x, y, t), M(x, Px, t), \\ M(y, Py, t), \frac{M(x,Px,t)M(y,Py,t)}{M(x,y,t)} \end{matrix}\right\}\right]$$

$x, y \in X$, and $\Omega: r_s(1 + i_2)e^{i_1\theta} \to r_s(1 + i_2)e^{i_1\theta}$ with $\Omega(t) > t, 0 < t < 1, r_s \in [0,1]$,

$\theta \in [0, \frac{\pi}{2}]$ Then P has a unique fixed point of X.

Proof: let $\sigma_0 \in X$ and a sequence $\{\sigma_n\}$ in X define by $\sigma_{n+1} = P\sigma_n, n = 0,1,2 \ldots$ then there are two possibilities are arise:

Possibilities 1: when $\sigma_n \neq \sigma_{n+1}$

Put $x = \sigma_n$ and $y = \sigma_{n+1}$ in inequality 3.1 then we have

$$M(P\sigma_n, P\sigma_{n+1}, ht) \gtrsim_{i_2} \Omega\left[Min\left\{\begin{matrix} M(\sigma_n, \sigma_{n+1}, t), M(\sigma_n, P\sigma_n, t), \\ M(\sigma_{n+1}, P\sigma_{n+1}, t), \frac{M(\sigma_n,P\sigma_n,t)M(\sigma_{n+1},P\sigma_{n+1},t)}{M(\sigma_n,\sigma_{n+1},t)} \end{matrix}\right\}\right]$$

$$M(\sigma_{n+1}, \sigma_{n+2}, ht) \gtrsim_{i_2} \Omega\left[Min\left\{\begin{matrix} M(\sigma_n, \sigma_{n+1}, t), M(\sigma_n, \sigma_{n+1}, t), \\ M(\sigma_{n+1}, \sigma_{n+2}, t), \frac{M(\sigma_n,\sigma_{n+1},t)M(\sigma_{n+1},\sigma_{n+2},t)}{M(\sigma_n,\sigma_{n+1},t)} \end{matrix}\right\}\right]$$

$$M(\sigma_{n+1}, \sigma_{n+2}, ht) \gtrsim_{i_2} \Omega\left[Min\left\{\begin{matrix} M(\sigma_n, \sigma_{n+1}, t), M(\sigma_n, \sigma_{n+1}, t), \\ M(\sigma_{n+1}, \sigma_{n+2}, t), M(\sigma_{n+1}, \sigma_{n+2}, t) \end{matrix}\right\}\right]$$

If $Min\left\{\begin{matrix} M(\sigma_n, \sigma_{n+1}, t), M(\sigma_n, \sigma_{n+1}, t), \\ M(\sigma_{n+1}, \sigma_{n+2}, t), M(\sigma_{n+1}, \sigma_{n+2}, t) \end{matrix}\right\} = M(\sigma_{n+1}, \sigma_{n+2}, t)$ then

$M(\sigma_{n+1}, \sigma_{n+2}, ht) \gtrsim_{i_2} M(\sigma_{n+1}, \sigma_{n+2}, t)$, which is contradiction.

Thus $Min\left\{\begin{matrix} M(\sigma_n, \sigma_{n+1}, t), M(\sigma_n, \sigma_{n+1}, t), \\ M(\sigma_{n+1}, \sigma_{n+2}, t), M(\sigma_{n+1}, \sigma_{n+2}, t) \end{matrix}\right\} = M(\sigma_n, \sigma_{n+1}, t)$ then

$$M(\sigma_{n+1}, \sigma_{n+2}, ht) \gtrsim_{i_2} \Omega[M(\sigma_n, \sigma_{n+1}, t)]$$

$$M(\sigma_{n+1}, \sigma_{n+2}, ht) \gtrsim_{i_2} M(\sigma_n, \sigma_{n+1}, t)$$

$$M(\sigma_n, \sigma_{n+1}, t) \gtrsim_{i_2} M\left(\sigma_{n-1}, \sigma_n, \frac{t}{h}\right)$$

$\gtrsim_{i_2} M\left(\sigma_{n-2}, \sigma_{n-1}, \frac{t}{h^2}\right)$

$\gtrsim_{i_2} M\left(\sigma_{n-3}, \sigma_{n-2}, \frac{t}{h^3}\right)$

$\gtrsim_{i_2} M\left(\sigma_0, \sigma_1, \frac{t}{h^n}\right)$

In general

$M(\sigma_n, \sigma_{n+m}, t) \gtrsim_{i_2} M\left(\sigma_n, \sigma_{n+m}, \frac{t}{h}\right) * \ldots.* M\left(\sigma_{n+m}, \sigma_{n+m-1}, \frac{t}{h}\right)$

$\gtrsim_{i_2} M\left(\sigma_0, \sigma_1, \frac{t}{h^n}\right) * \ldots.* M\left(\sigma_0, \sigma_1, \frac{t}{h^{n+m}}\right)$

Making $n \to \infty$ then

$\lim_{n\to\infty} M(\sigma_n, \sigma_{n+m}, t) \gtrsim_{i_2} \lim_{n\to\infty} \left\{M\left(\sigma_n, \sigma_{n+m}, \frac{t}{h}\right) * \ldots.* M\left(\sigma_{n+m}, \sigma_{n+m-1}, \frac{t}{h}\right)\right\}$

But $\lim_{t\to\infty} M(x, y, t) = (1 + i_2)e^{i_1\theta}$

$\lim_{n\to\infty} M(\sigma_n, \sigma_{n+m}, t) \gtrsim_{i_2} (1 + i_2)e^{i_1\theta} * (1 + i_2)e^{i_1\theta} * \ldots.* (1 + i_2)e^{i_1\theta}$

$\lim_{n\to\infty} M(\sigma_n, \sigma_{n+m}, t) \gtrsim_{i_2} (1 + i_2)e^{i_1\theta}$

We observe that $\{\sigma_n\}$ is Cauchy sequence in X. since X is complete, there exists some $a \in X$ such that $\sigma_n \to \sigma$ as n $\to \infty$. Implying thereby the convergence of $\{\sigma_n\}$ and $\{\sigma_{n+1}\}$ being sub-sequences of the convergent sequence $\{\sigma_n\}$. Then $\sigma_n \to$ and $\sigma_{n+1} \to \sigma$ as $n \to \infty$.

Now we shall show that σ is a fixed point of P .

Now consider

$M(P\sigma, \sigma, t) \gtrsim_{i_2} M\left(P\sigma, \sigma_{n+1}, \frac{t}{2}\right) * M(\sigma_{n+1}, \sigma, \frac{t}{2})$

$$\gtrsim_{i_2} \left\{\Omega\left[Min\left\{\begin{array}{c} M\left(\sigma, \sigma_n, \frac{t}{2h}\right), M\left(\sigma, P\sigma, \frac{t}{2h}\right), \\ M\left(\sigma_n, P\sigma_n, \frac{t}{2h}\right), \frac{M\left(\sigma, P\sigma, \frac{t}{2h}\right) M\left(\sigma_n, P\sigma_n, \frac{t}{2h}\right)}{M\left(\sigma, \sigma_n, \frac{t}{2h}\right)} \end{array}\right\}\right]\right\}$$

$* M(\sigma_{n+1}, \sigma, \frac{t}{2})$

$$\gtrsim_{i_2} \left\{\Omega\left[Min\left\{\begin{array}{c} M\left(\sigma, \sigma, \frac{t}{2h}\right), M\left(\sigma, P\sigma, \frac{t}{2h}\right), \\ M\left(\sigma, P\sigma, \frac{t}{2h}\right), \frac{M\left(\sigma, P\sigma, \frac{t}{2h}\right) M\left(\sigma, P\sigma, \frac{t}{2h}\right)}{M\left(\sigma, \sigma, \frac{t}{2h}\right)} \end{array}\right\}\right]\right\}$$

$M(\sigma, \sigma, \frac{t}{2})$

$$\gtrsim_{i_2} \left\{\Omega\left[Min\left\{\begin{array}{c} (1+i_2)e^{i_1\theta}, M\left(\sigma, P\sigma, \frac{t}{2h}\right), \\ M\left(\sigma, P\sigma, \frac{t}{2h}\right), \frac{M\left(\sigma, P\sigma, \frac{t}{2h}\right) M\left(\sigma, P\sigma, \frac{t}{2h}\right)}{(1+i_2)e^{i_1\theta}} \end{array}\right\}\right]\right\}$$

$(1 + i_2)e^{i_1\theta}$

$M(P\sigma, \sigma, t) \gtrsim_{i_2} M\left(\sigma, P\sigma, \frac{t}{2h}\right)$

Which gives $M(P\sigma, \sigma, t) = (1 + i_2)e^{i_1\theta}$ because $h \in (0, \frac{1}{2})$

$\Rightarrow P\sigma = \sigma$

Thus a is a fixed point of P. Hence a is a common fixed point of mappings P.

Uniqueness: let $b \in X$ be another common fixed point of mappings P such that $a \neq b$ so $Pa = Pb$

Then by inequality 3.1

$M(Pa, Pb, ht)$

$$\gtrsim_{i_2} \Omega\left[Min\left\{\begin{array}{c} M(a, b, t), M(a, Pa, t), \\ M(b, Pb, t), \frac{M(a,Pa,t)M(b,Pb,t)}{M(a,b,t)} \end{array}\right\}\right]$$

$M(a, b, ht)$

$$\gtrsim_{i_2} \Omega\left[Min\left\{\begin{array}{c} M(a, b, t), (1 + i_2)e^{i_1\theta}, \\ (1 + i_2)e^{i_1\theta}, \frac{(1+i_2)e^{i_1\theta}(1+i_2)e^{i_1\theta}}{M(a,b,t)} \end{array}\right\}\right]$$

$M(a, b, ht)$

$$\gtrsim_{i_2} \Omega\left[Min\left\{\begin{array}{c} M(a, b, t), (1 + i_2)e^{i_1\theta}, \\ (1 + i_2)e^{i_1\theta}, \frac{(1+i_2)e^{i_1\theta}(1+i_2)e^{i_1\theta}}{M(a,b,t)} \end{array}\right\}\right]$$

Since $M(a, b, t) \in r_s(1 + i_2)e^{i_1\theta}$ with $\Omega(t) > t$, , $r_s \in [0,1], \theta \in [0, \frac{\pi}{2}]$

$M(a, b, ht) \gtrsim_{i_2} \Omega[M(a, b, t)]$

$M(a, b, ht) \gtrsim_{i_2} M(a, b, t)$

By lemma 3.11 we get $a = b$

Thus a is the unique common fixed point of P.

Possibilities 2: when $\sigma_n \neq \sigma_{n+1}$

We observe that $\{\sigma_n\}$ is a constant sequence and so convergent. This concludes the proof.

4. Theorem (3.2): A bicomplex valued fuzzy metric space $(X, M, *)$ with $\lim_{t\to\infty} M(x,y,t) = (1+i_2)e^{i_1\theta}$. let $P, Q, R, T: X \to X$ be a mapping satisfying and the pairs (P,R) *and* (Q,T) be occasionally weakly compatible if there exist $h \in (0,1)$ such that

$$M(Px,Qy,ht) \succsim_{i_2} Min\begin{Bmatrix} M(Rx,Ty,t), M(Rx,Px,t), \\ M(Rx,Qy,t), M(Qy,Ty,t) \end{Bmatrix}$$

$x, y \in X$, and $0 < t < 1$ then there exist a unique common fixed point of P, Q, R *and* T.

Proof: Since the pairs (P,R) *and* (Q,T) be occasionally weakly compatible, so there are points $x, y \in X$ such that $Px = Rx$ *and* $Qy = Ty$.

Now we shall claim $Px = Qy$, if not then by inequality (3.2) we have

$$M(Px,Qy,ht) \succsim_{i_2} Min\begin{Bmatrix} M(Rx,Ty,t), \ M(Px,Px,t), \\ M(Px,Qy,t), \ M(Ty,Ty),t) \end{Bmatrix}$$

$$M(Px,Qy,ht) \succsim_{i_2} Min\begin{Bmatrix} M(Rx,Ty,t), \ (1+i_2)e^{i_1\theta}, \\ M(Px,Qy,t), \ (1+i_2)e^{i_1\theta} \end{Bmatrix}$$

$$M(Px,Qy,ht) \succsim_{i_2} M(Px,Qy,t)$$

Therefore in view of lemma 2.11, we have $Px = Qy$.

$i.e \ Px = Rx = Qy = Ty$.

Suppose that there is another coincidence point $u \in X$ such that $Pu = Ru$.

Then by inequality (3.2)

$$M(Pu,Qy,ht) \succsim_{i_2} Min\begin{Bmatrix} M(Ru,Ty,t), M(Ru,Pu,t), \\ M(Ru,Qy,t), M(Qy,Ty),t) \end{Bmatrix}$$

$$M(Pu,Qy,ht) \succsim_{i_2} Min\begin{Bmatrix} M(Pu,Qy,t), M(Pu,Pu,t), \\ M(Pu,Qy,t), M(Qy,Qy,t) \end{Bmatrix}$$

$$M(Pu,Qy,ht) \succsim_{i_2} Min\begin{Bmatrix} M(Pu,Qy,t), (1+i_2)e^{i_1\theta}, \\ M(Pu,Qy,t), (1+i_2)e^{i_1\theta} \end{Bmatrix}$$

$$M(Pu,Qy,ht) \succsim_{i_2} M(Pu,Qy,t)$$

Therefore in view of lemma 2.11, we have $Pu = Qy$.

$i.e \ Pu = Ru = Qy = Ty$.

So $Px = Pu$ and $w = Px = Rx$ is the unique point of coincidence of pair (P,R).

Similarly we can show if there is a unique common fixed point $z \in X$ in pair (Q,T).

Such that $u = Qu = Tu$.

Suppose that $w \neq u$ then by inequality (3.2)

$$M(Pw,Qu,ht) \succsim_{i_2} Min\begin{Bmatrix} M(Rw,Tu,t), M(Rw,Pw,t), \\ M(Rw,Qu,t), M(Qu,Tu,t) \end{Bmatrix}$$

$$M(w,u,ht) \succsim_{i_2} Min\begin{Bmatrix} M(w,u,t), M(w,w,t), \\ M(w,u,t), M(u,u,t) \end{Bmatrix}$$

$$M(w,u,ht) \succsim_{i_2} Min\begin{Bmatrix} M(w,u,t), (1+i_2)e^{i_1\theta}, \\ M(w,u,t), (1+i_2)e^{i_1\theta} \end{Bmatrix}$$

$$M(w,u,ht) \succsim_{i_2} M(w,u,t)$$

Therefore we have $w = u$ then by lemma 2.11, u is a common fixed point of P, Q, R *and* T.

Uniqueness: let v be the another common fixed point of P, Q, R *and* T such that $u \neq v$

Then by inequality (3.2)

$$M(Pu,Qv,ht) \succsim_{i_2} Min\begin{Bmatrix} M(Ru,Tv,t), M(Ru,Pu,t), \\ M(Ru,Qv,t), M(Qv,Tv,t) \end{Bmatrix}$$

$$M(u,v,ht) \succsim_{i_2} Min\begin{Bmatrix} M(u,v,t), M(u,u,t), \\ M(u,v,t), M(v,v,t) \end{Bmatrix}$$

$$M(u,v,ht) \succsim_{i_2} Min\begin{Bmatrix} M(u,v,t), (1+i_2)e^{i_1\theta}, \\ M(u,v,t), (1+i_2)e^{i_1\theta} \end{Bmatrix}$$

$$M(u,v,ht) \succsim_{i_2} M(u,v,t)$$

By lemma 2.11, we obtain $u = v$.

Therefore $u = v$ is a unique common fixed point of P, Q, R *and* T.

Example 7: let $X = \{0\} \cup N$, *Consider* $d(x,y) = |x-y|$ be a metric space, Let $m * n = min\{m, n\}$. $for\ x, y\ on\ X$. define, $for\ all\ m, n \in r_s(1 + i_2)e^{i_1\theta}$,

Where, $r_s \in [0,1]$, $\theta \in \left[0, \frac{\pi}{2}\right]$, for each $t > 0, x, y \in X$.

Define $\mathcal{M}(x, y, t) = (1 + i_2)e^{i_1\theta} \frac{t}{t+d(x,y)}$ with $\lim_{t\to\infty} M(x, y, t) = (1 + i_2)e^{i_1\theta}$.

Now we define the self maps P on X by

$Px = \frac{x}{5}$, for all $x \in X$.

Set $h = \frac{1}{3}$ if $x \neq y$ then

$$\mathcal{M}\left(Px, Py, \frac{t}{2}\right) = (1 + i_2)e^{i_1\theta} \frac{\frac{t}{3}}{\frac{t}{3} + |Px - Py|}$$

$$= (1 + i_2)e^{i_1\theta} \frac{t}{t + 3\left|\frac{x}{5} - \frac{y}{5}\right|}$$

$$M(Px, Py, ht) \gtrsim_{i_2} \Omega \left[Min \left\{ \begin{matrix} M(x, y, t), M(x, Px, t), \\ M(y, Py, t), \frac{M(x,Px,t)M(y,Py,t)}{M(x,y,t)} \end{matrix} \right\} \right]$$

Therefore, the maps P satisfies the theorem 3.1 for h= 1/3. Therefore x = 0 is unique common fixed point.

III. CONCLUSION

In many scientific domains, fixed point theory has several applications. Common fixed point findings in bicomplex valued fuzzy metric spaces are established in this research. Using sometimes weakly compatible (owc) conditions, our results enhanced, expanded, and generalized some findings in the literature. These findings can be applied to the solution of LPP in dynamic programming, quantum mechanics, image processing, control systems, machine learning, robotics, digital problems, economics, and other fields. These uses demonstrate how complex valued fuzzy metric spaces may be used to handle a wide range of data kinds and uncertainty, making them an invaluable tool in many engineering and scientific fields.

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